

# ON CLASSICAL UNIFORMIZATION THEOREMS FOR HIGHER DIMENSIONAL COMPLEX KLEINIAN GROUPS

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**ABSTRACT.** In this article we show that Bers' simultaneous uniformization as well as the K  be's retrosection theorem are not longer true for discrete groups of projective transformations acting on the complex projective space.

## INTRODUCTION

The uniformization theorems of Riemann surfaces plays a mayor role in one dimensional complex dynamics, its study has its roots in the work of Poincar   and K  be (see [16–18, 21]), the Riemann uniformization theorem asserts that any simply connected Riemann surface is biholomorphic either to the sphere  $\hat{\mathbb{C}}$  the plane  $\mathbb{C}$  or the disc  $\mathbb{D}$ , in 1910, K  be (see [19]) “improved” this ideas in his Retrosection theorem proving that for any closed Riemann surface there is a Schottky group such that the associated domain of discontinuity uniformizes the surface, later in 1960 Bers (see [1]) with his simultaneous uniformization theorem gives a “generalisation” of the retrosection theorem, by asserting that for any two closed Riemann surfaces of the same genus, there is a quasi-Fuchsian group that uniformizes the two surfaces.

Around 1990, J. Seade and A. Verjovsky (see [22]) introduced the concept of complex Kleinian group as a discrete subgroup of  $PSL(n, \mathbb{C})$  acting on the projective complex space with an open subset where the action of the group is properly discontinuous, a very important family of complex Kleinian groups are the subgroups of isometries of the complex hyperbolic space (see [12]). In view of this groups one natural question arises, what about the uniformization of higher dimensional complex manifolds in terms of complex Kleinian groups?, in this case we have that the geometry of higher dimensional complex manifolds is much more diverse, to exemplify this let us consider the following facts:

- (1) The complex manifold  $\mathbb{P}_{\mathbb{C}}^n \times \mathbb{P}_{\mathbb{C}}^n$  is a simply connected complex manifold that does not admit a complex projective structure, so if we want to ask for the uniformization of higher dimensions in terms of complex Kleinian groups we must require that the manifolds admit a complex projective structure (this is going to be a big first difference with the one dimensional case).
- (2) In [11] and [23] they explain the Smilie's construction of a torus with a complex projective structure that is not complete, that is the manifold cannot be realised as the quotient of an open set of the complex projective line and a discrete subgroup that acts discontinuously on it. Once again,

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when asking for uniformization results we must assure that the manifolds that we will study have a complex projective structure which is complete.

- (3) Last important fact to remember is that in the higher dimensional setting there are several simply connected domains of  $\mathbb{P}_{\mathbb{C}}^n$  which are not biholomorphically equivalent. Moreover, some of this domains arise as connected components in the equicontinuity region of complex Kleinian groups, see [6–8].

Taking in count this facts we see that is fully expected that classical uniformization theorems fail in the higher dimensional setting, in this article are able to prove:

**Theorem 0.1.** *There is not an analogue of Bers simultaneous uniformization theorem for groups of  $PSL(n+1, \mathbb{C})$  acting on  $\mathbb{P}_{\mathbb{C}}^n$ , where  $n \geq 2$ .*

This result was proven for  $n = 2$  in [7]. We also show that the algebraic and geometric higher dimensional analog of the K oebe’s retrosection theorem is false, here algebraic means that the fundamental group of a manifold has a representation as a purely loxodromic free discrete group and geometric means that the manifold can be realised as a quotient by a Schottky group.

**Theorem 0.2.** *The geometric and algebraic version of K oebe’s retrosection is not longer true for groups of  $PSL(n+1, \mathbb{C})$  acting on  $\mathbb{P}_{\mathbb{C}}^n$ , where  $n \geq 2$ .*

A weaker version of this result was essentially proven for  $n$  even in [3].

The paper is organised as follows: in section 1 we introduce the terminology used along the article, in section 2 we introduce the notion of Schottky like groups and prove a technical lemma concerning its dynamic which will be useful in the last section, finally in section 3 we provide full proofs of the main results of this paper.

## 1. PRELIMINARIES

The complex projective space  $\mathbb{P}_{\mathbb{C}}^n$  is defined as:  $\mathbb{P}_{\mathbb{C}}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ , where  $\mathbb{C}^*$  acts by the usual scalar multiplication. If  $[\ ] : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}_{\mathbb{C}}^n$  is the quotient map, then a non-empty set  $H \subset \mathbb{P}_{\mathbb{C}}^n$  is said to be a projective subspace if there is a  $\mathbb{C}$ -linear subspace  $\tilde{H}$  such that  $[\tilde{H} \setminus \{0\}] = H$ . In this article,  $e_1, \dots, e_{n+1}$  will denote the standard basis for  $\mathbb{C}^{n+1}$ . Given a set of points  $P$  in  $\mathbb{P}_{\mathbb{C}}^n$ , we define:

$$Span(P) = \bigcap \{l \subset \mathbb{P}_{\mathbb{C}}^n \mid l \text{ is a projective subspace containing } P\}.$$

The group of projective automorphisms of  $\mathbb{P}_{\mathbb{C}}^n$  is  $PSL(n+1, \mathbb{C}) = GL(n+1, \mathbb{C})/\mathbb{C}^*$ , where  $\mathbb{C}^*$  acts by the usual scalar multiplication,  $PSL(n+1, \mathbb{C})$  is a Lie group whose elements are called projective transformations. We denote by  $[[\ ]] : GL(n+1, \mathbb{C}) \rightarrow PSL(n+1, \mathbb{C})$  the quotient map. Given  $\gamma \in PSL(n+1, \mathbb{C})$ , we say that  $\tilde{\gamma} \in GL(n+1, \mathbb{C})$  is a lift of  $\gamma$  if  $[[\tilde{\gamma}]] = \gamma$ .

Now consider the following Hermitian form  $\langle \cdot, \cdot \rangle : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ , given by:

$$\langle z, w \rangle = z_1 \overline{w_1} + \dots + z_n \overline{w_n} - z_{n+1} \overline{w_{n+1}}$$

We set

$$U(1, n) = \{g \in GL(n+1, \mathbb{C}) : \langle g(z), g(w) \rangle = \langle z, w \rangle\}.$$

The respective projectivization  $PU(1, n)$  preserves the unitary complex ball:

$$\mathbb{H}_{\mathbb{C}}^n = \{[w] \in \mathbb{P}_{\mathbb{C}}^n \mid \langle w, w \rangle < 0\}$$

Elements in  $PU(1, n)$  splits into 3 types according to the position of its fixed points in  $\mathbb{H}_{\mathbb{C}}^n$ , more precisely:

**Definition 1.1.** Let  $\gamma \in PU(1, n)$ , then  $\gamma$  is called

- (1) elliptic if  $\gamma$  has a fixed point in  $\mathbb{H}_{\mathbb{C}}^n$ .
- (2) parabolic if  $\gamma$  has exactly one fixed point in  $\partial\mathbb{H}_{\mathbb{C}}^n$ .
- (3) loxodromic if  $\gamma$  has exactly two fixed point in  $\partial\mathbb{H}_{\mathbb{C}}^n$ .

Given a group  $\Gamma \subset PU(1, n)$ , we define the following notion of limit set due to Chen and Greenberg, see [9].

**Definition 1.2.** Let  $\Gamma \subset PU(1, n)$ , then  $\Lambda_{CG}(\Gamma)$  is to be defined as  $\overline{\Gamma x} \cap \partial\mathbb{H}_{\mathbb{C}}^n$ , where  $x \in \mathbb{H}_{\mathbb{C}}^n$  is any point.

As is the Fuchsian groups case, it is customary to show that  $\Lambda_{CG}(\Gamma)$  does not depend on the choice of  $x$  and  $\Lambda_{CG}(\Gamma)$  has either 1, 2 or infinite points. A group is said to be non-elementary if  $\Lambda_{CG}(\Gamma)$  has infinite points. In the following, given a projective subspace  $P \subset \mathbb{P}_{\mathbb{C}}^n$  we will define

$$P^{\perp} = [\{w \in \mathbb{C}^{n+1} \mid \langle w, v \rangle = 0 \text{ for all } v \in [P]^{-1}\} \setminus \{0\}].$$

**Lemma 1.3.** Let  $\gamma \in PU(1, n)$  be a loxodromic element and  $a, r \in \partial\mathbb{H}_{\mathbb{C}}^n$  be the attracting and repelling points of  $\gamma$  respectively, thus

- (1) We have  $\gamma^n \xrightarrow{m \rightarrow \infty} a$  uniformly on compact sets of  $\mathbb{P}_{\mathbb{C}}^n \setminus r^{\perp}$ .
- (2) We have  $\gamma^{-n} \xrightarrow{m \rightarrow \infty} r$  uniformly on compact sets of  $r^{\perp} \setminus a^{\perp}$ .
- (3) The transformation  $\gamma$  restricted to  $r^{\perp} \cap a^{\perp}$  is conjugate to an element of  $PU(n-1)$  acting on  $\mathbb{P}_{\mathbb{C}}^{n-1}$ .

A complex hyperbolic manifold is the quotient of a open subset of the complex hyperbolic space and a discrete subgroup of  $PU(1, n)$ . Mok-Young and Klingler showed independently that for a complex hyperbolic manifold with finite volume there is a unique complex projective structure compatible with the complex hyperbolic structure, this result will be crucial central in our discussion, see [14, 20], for sake of completeness here we write down the result.

**Theorem 1.4** (Mok-Young, Klingler). Let  $\Gamma \subset PU(1, n)$  be a discrete group such that  $M = \mathbb{H}_{\mathbb{C}}^n / \Gamma$  is manifold of finite volume, then  $M$  has only one projective structure compatible with the complex exstructure.

This is very deep result which has several dynamical consequences some of which we will see later.

## 2. SCHOTTKY LIKE GROUPS

Schottky groups are the “simplest” examples of Kleinian groups and enjoy very interesting properties, unfortunately they are always realizable in the higher dimensional setting (see [3]) in a “usual” way. For this reason let us introduce here a weaker form of Schottky groups, compare with [10].

**Definition 2.1.** Let  $\Sigma \subset PSL(n+1, \mathbb{C})$  a finite set which is symmetric (i. e.  $a^{-1} \in \Sigma$  for all  $a \in \Sigma$ ) and  $\{A_a\}_{a \in \Sigma}$  a family of compact non-empty pairwise disjoint subsets of  $\mathbb{P}_{\mathbb{C}}^n$  such that for each  $a \in \Sigma$  we have

$$\bigcup_{b \in \Sigma \setminus \{a^{-1}\}} a(A_b) \subsetneq A_a.$$

The group  $\Gamma$  generated by  $\Sigma$  is called Schottky like group. We define a “kind” of limit set for  $\Gamma$  as follows:

$$\Lambda_S(\Gamma) = \overline{\{y \in \mathbb{P}_{\mathbb{C}}^n \mid \exists(\phi_m) \subset \Sigma, (y_n) \subset A_{\phi_0} : \phi_{j+1}\phi_j \neq Id, \phi_m \circ \dots \circ \phi_1(y_m) \xrightarrow{m \rightarrow \infty} y\}}$$

Clearly every Schottky like group is a free, finitely generated and discrete.

**Example 2.2.** Every Schottky group of  $PSL(2, \mathbb{C})$  acting on  $\mathbb{P}_{\mathbb{C}}^1$  is a Schottky like group.

The following is a straightforward result which will be used later, see [3, 4] for a proof.

**Lemma 2.3.** *Let us consider the cyclic group  $\Gamma \subset PSL(n+1, \mathbb{C})$  generated by the element*

$$\gamma = \begin{bmatrix} A & \\ & B \end{bmatrix}$$

where  $A$  is a  $k \times k$  diagonalisable matrix with unitary proper values and  $B$  is a  $(n+1-k) \times (n+1-k)$ -Jordan block whose proper value is  $Det(A)^{-(n+1-k)^{-1}}$ . Then

- (1) *If  $x \in \mathbb{P}_{\mathbb{C}}^n \setminus Span(e_1, \dots, e_k)$ , then the set of accumulation points of  $\Gamma x$  is  $e_1$ .*
- (2) *If  $x \in Span(e_1, \dots, e_k)$ , then  $x$  belongs to the set of accumulation points of  $\Gamma x$ .*

**Lemma 2.4.** *Let  $\Gamma \subset PU(1, n)$  be a Schottky like group, thus*

- (1) *The group  $\Gamma$  is purely loxodromic group.*
- (2) *We have*

$$\Lambda_{CG}(\Gamma) \subset \Lambda_S(\Gamma).$$

- (3) *The set  $\Lambda_{CG}(\Gamma)$  is disconnected.*

*Proof.* Let us show (1). Clearly, it will be enough to show that every generator is loxodromic. Let  $\gamma \in \Gamma$  be a generator, since  $\Gamma$  is free, we deduce that  $\gamma$  is either parabolic or loxodromic. Let us assume that  $\gamma$  is parabolic, since  $\gamma(A_\gamma) \subsetneq A_\gamma$  and  $\gamma^{-1}(A_{\gamma^{-1}}) \subsetneq A_{\gamma^{-1}}$  for some pairwise disjoint, non-empty compact sets of  $\mathbb{P}_{\mathbb{C}}^n$  we deduce that  $\gamma$  has at least two fixed points. Therefore  $\gamma$  has a lift  $\tilde{\gamma} \in SL(n+1, \mathbb{C})$  whose normal Jordan form is, see [4, 9]:

$$\gamma = \begin{bmatrix} A & \\ & B \end{bmatrix}$$

where  $A$  is a  $k \times k$  diagonalisable matrix with unitary proper values and  $B$  is a  $(n+1-k) \times (n+1-k)$ -Jordan block whose proper value is  $Det(A)^{-(n+1-k)^{-1}}$ . Finally, let  $x \in A_\gamma$  and  $y \in A_{\gamma^{-1}}$  be fixed point, then by Lemma 2.3 we conclude that  $x = y = e_{k+1}$ , which is a contradiction.

The proof of (2) goes as follows. Since  $\Gamma$  is free we deduce that  $\Gamma$  is non-elementary, see [13]. Therefore, it will be enough to show that for every generator  $Fix(\gamma) \cap \partial \mathbb{H}_{\mathbb{C}}^n \cap \Lambda_S(\Gamma) \neq \emptyset$ . On the contrary, let us assume that  $Fix(\gamma) \cap \partial \mathbb{H}_{\mathbb{C}}^n \cap \Lambda_S(\Gamma) = \emptyset$  for every generator. Let  $\gamma_1, \gamma_2 \in \Gamma$  be a generators satisfying  $\gamma_1 \gamma_2 \neq Id$ . For each  $i = 1, 2$ , let  $A_{\gamma_i}$  be the generating set of  $\gamma_i$ , since  $\gamma_1$  is loxodromic we can consider  $a, r \in \partial \mathbb{H}_{\mathbb{C}}^n$  the attracting and repelling fixed points of  $\gamma_1$ , respectively. Therefore  $A_{\gamma_2} \subset a^\perp \cap r^\perp$ , in consequence  $A_{\gamma_2} \subset A_{\gamma_1}$ . Since  $\gamma_1$  restricted to  $r^\perp \cap a^\perp$  is elliptic, which is a contradiction.

Finally observe that part (3) is trivial.  $\square$

## 3. PROOF OF THE MAIN THEOREMS

**Proof of theorem 0.1.** Let  $M$  be a compact complex manifold such that  $M = \mathbb{H}_{\mathbb{C}}^n/\Gamma$ , where  $n \geq 2$  and  $\Gamma \subset PU(1, n)$  is a discrete group. Let us consider the manifold  $M \sqcup M$  and let us assume that there is a group  $G \subset PSL(n+1, \mathbb{C})$  and a  $G$ -invariant open set  $U \subset \mathbb{P}_{\mathbb{C}}^n$  satisfying  $M \sqcup M = U/G$ . By Theorem 1.4, we deduce that  $G = \Gamma$ , up to projective conjugation. On the other hand, by the main theorem in [5], we know that  $\mathbb{H}_{\mathbb{C}}^n$  is the largest open set of  $\mathbb{P}_{\mathbb{C}}^n$  on which  $\Gamma$  acts properly discontinuously, which is a contradiction.  $\square$

**Proof of theorem 0.2.** [Geometric version] Let  $M$  be a compact complex manifold such that  $M = \mathbb{H}_{\mathbb{C}}^n/\Gamma$ , where  $n \geq 2$  and  $\Gamma \subset PU(1, n)$  is a discrete group. Let us assume that there is  $G \subset PSL(n+1, \mathbb{C})$  a Schottky like group and a  $G$ -invariant open set  $U \subset \mathbb{P}_{\mathbb{C}}^n$   $M = U/G$ . By Theorem 1.4 we deduce that  $G = \Gamma$ , up to projective conjugation. Finally observe that  $\Lambda_{CG}(\Gamma) = \partial\mathbb{H}_{\mathbb{C}}^n$ , since  $M$  is compact, however, this contradicts Lemma 2.4.  $\square$

**Proof of theorem 0.2.** [Algebraic version] Let  $M$  be a compact complex manifold such that  $M = \mathbb{H}_{\mathbb{C}}^n/\Gamma$ , where  $n \geq 2$  and  $\Gamma \subset PU(1, n)$  is a discrete group. Let us assume that there is  $G \subset PSL(n+1, \mathbb{C})$  a discrete, purely loxodromic free group and a  $G$ -invariant open set  $U \subset \mathbb{P}_{\mathbb{C}}^n$  satisfying  $M = U/G$ . By Theorem 1.4 we deduce that  $G = \Gamma$ , up to projective conjugation. Since  $M$  is compact, we get that the Cayley graph  $\Delta(\Gamma)$  of  $\Gamma$  is quasi-isometric to  $\mathbb{H}_{\mathbb{C}}^n$ , see [2], in consequence the Gromov boundaries  $\partial\Delta(\Gamma)$  and  $\partial\mathbb{H}_{\mathbb{C}}^n$  are homeomorphic, see [2]. This is a contradiction since it is well known, see [2], that  $\partial\Delta(\Gamma)$  is a Cantor set while  $\partial\mathbb{H}_{\mathbb{C}}^n$  is the  $2n-1$ -sphere.  $\square$

As we have seen, the proofs of the results are in terms of compact complex hyperbolic manifolds a more interesting question is to determine sufficient conditions for the uniformization theorems remain valid in the higher dimensional case, see for example [15] for results concerning compact complex projective surfaces.

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